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Isotropic Kähler hyperbolic twistor spaces

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Abstract

In this paper we study two natural indefinite almost Hermitian structures on the hyperbolic twistor space of a four-manifold endowed with a neutral metric. We show that only one of these structures can be isotropic Kähler and obtain the precise geometric conditions on the base manifold ensuring this property.

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1. Introduction

Let (M, J, g) be an almost Hermitian manifold with an almost complex structure J and compatible Riemannian metric g, i.e. $g(X, Y) = g(JX, JY)$. If J is parallel with respect to the Levi-Civita connection ∇ of g, the structure is Kähler and this can be recognised by the vanishing of the square norm of ∇J, i.e. $\|\nabla J\|^2 = 0$, equivalently by $\|\nabla \Omega\|^2 = 0$, where Ω is the fundamental two-form of the almost Hermitian structure. However for indefinite metrics this is not true, i.e. in general the vanishing of the square norm $\|\nabla J\|^2$ does not always imply the Kähler condition, $\nabla J = 0$. Thus an indefinite almost Hermitian structure is said to be *isotropic Kähler* if $\|\nabla J\|^2 = 0$ and *indefinite Kähler* if $\nabla J = 0$. Although there are many known examples of indefinite Kähler structures (see, e.g. [2,5,11]) the

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first examples of non-Kähler isotropic Kähler structures have been recently constructed by Garcia-Rio and Matsushita [7]. These examples are non-integrable and are related to certain Engel structures on $\mathbb{R}^4$ and their compact quotients.

In this paper we give a class of integrable six-dimensional examples. These arise as certain indefinite Hermitian structures on hyperbolic twistor spaces over four-dimensional manifolds [3]. One of the key features of our study is the fact that the base manifolds are endowed with neutral metrics, i.e. semi-Riemannian metrics of signature $(2, 2)$. This setting naturally arises in $N = 2$ string theory where the additional structure of two local supersymmetries on a worldsheet leads to considering neutral Kähler metrics. We refer to [10] for a fascinating discussion.

In Section 2 we review the theory of hyperbolic twistor spaces over four-dimensional manifolds with neutral metrics and their two natural indefinite almost Hermitian structures. In Section 3 we compute the square norm of the covariant derivatives of their fundamental two-forms in terms of the curvature of the base manifold. Then in Section 4 we prove that only one of these indefinite almost Hermitian structures can be isotropic Kähler and obtain the precise geometric conditions on the base manifold ensuring this property. Finally in Section 5 we construct two-parameter families of neutral left-invariant metrics on some four-dimensional Lie groups whose hyperbolic twistor spaces are indefinite Hermitian and isotropic Kähler but not indefinite Kähler.

2. Hyperbolic twistor spaces over four-dimensional manifolds

Let M be an oriented four-dimensional manifold with a neutral metric g, i.e. a pseudo-Riemannian metric of signature $(2, 2)$, and $\mathbf{e}_1, \ldots, \mathbf{e}_4$ a local orthonormal frame with $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$ giving the orientation. The metric g induces a metric on bundle of bivectors, $\wedge^2 TM$, by

$$g(\mathbf{e}_i \wedge \mathbf{e}_j, \mathbf{e}_k \wedge \mathbf{e}_l) = \frac{1}{2} \begin{vmatrix} \varepsilon_i \delta_{ik} & \varepsilon_i \delta_{il} \\ \varepsilon_j \delta_{jk} & \varepsilon_j \delta_{jl} \end{vmatrix}, \quad \varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = -1.$$

The Hodge star operator of the neutral metric acts as an involution on $\wedge^2 TM$ and is given by

$$*(\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_3 \wedge \mathbf{e}_4, \qquad *(\mathbf{e}_1 \wedge \mathbf{e}_3) = \mathbf{e}_2 \wedge \mathbf{e}_4, \qquad *(\mathbf{e}_1 \wedge \mathbf{e}_4) = -\mathbf{e}_2 \wedge \mathbf{e}_3.$$

Let $\wedge^-$ and $\wedge^+$ denote the subbundles of $\wedge^2 TM$ determined by the corresponding eigenvalues of the Hodge star operator. The metrics induced on $\wedge^-$ and $\wedge^+$ have signature $(+, -, -)$.

Setting

$$s_1 = \mathbf{e}_1 \wedge \mathbf{e}_2 - \mathbf{e}_3 \wedge \mathbf{e}_4, \qquad \bar{s}_1 = \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4,$$

$$s_2 = \mathbf{e}_1 \wedge \mathbf{e}_3 - \mathbf{e}_2 \wedge \mathbf{e}_4, \qquad \bar{s}_2 = \mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_4,$$

$$s_3 = \mathbf{e}_1 \wedge \mathbf{e}_4 + \mathbf{e}_2 \wedge \mathbf{e}_3, \qquad \bar{s}_3 = \mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3,$$

$\{s_1, s_2, s_3\}$ and $\{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$ are local oriented orthonormal frames for $\wedge^-$ and $\wedge^+$, respectively.

An *almost paraquaternionic structure* on a C^∞ manifold M is a rank 3-subbundle E of the endomorphisms bundle End(TM) which locally is spanned by a triple $\{J_1, J_2, J_3\}$, where J_1 is an almost complex structure, J_2 an almost product structure such that $J_1J_2 + J_2J_1 = 0$ and $J_3 = J_1J_2$. J_3 is a second almost product structure which also anti-commutes with J_1 and J_2; in particular $\{J_1, J_2, J_3\}$ is an almost quaternionic structure of the second kind in the sense of Libermann [9]. An almost paraquaternionic manifold of dimension $4n \geq 8$ and neutral metric g is said to be *paraquaternionic Kähler* if the bundle E is parallel with respect to the Levi-Civita connection D of g. In dimension 4 this is not a restriction and the four-dimensional analogue of a paraquaternionic Kähler manifold is a neutral, Einstein, self-dual manifold.

We can now identify $\wedge^2 TM$ with the bundle of skew-symmetric endomorphisms of TM by the correspondence that assigns to each $\sigma \in \wedge^2 TM$ the endomorphism K_σ on T_pM, $p = \pi(\sigma)$, defined by

$$g(K_\sigma X, Y) = 2g(\sigma, X \wedge Y); \quad X, Y \in T_pM.$$

Now the bundle $E = \wedge^-$ defines an almost paraquaternionic structure on M, the local endomorphisms $\{J_1, J_2, J_3\}$ spanning E being $J_1 = K_{s_1}$, $J_2 = K_{s_2}$, $J_3 = K_{s_3}$.

We can now define the twistor space $\mathcal{Z}$ as given in [3]. We first observe that if

$$j = y_1J_1 + y_2J_2 + y_3J_3,$$

then j is an almost complex structure on M if and only if

$$-y_1^2 + y_2^2 + y_3^2 = -1.$$

The *hyperbolic twistor space* $\pi : \mathcal{Z} \to M$ is then the hypersurface of E defined by this equation. In particular the fibres of $\mathcal{Z}$ are these hyperboloids and the reader is encouraged to think of the $y_1 > 0$ branch as one of the standard models of the hyperbolic plane.

Define a pseudo-Riemannian metric on $\mathcal{Z}$ by

$$h_t = \pi^* g + t\langle , \rangle, \quad t \neq 0,$$

where $\langle , \rangle$ is the negative of the restriction of induced metric on E to the fibres. When $t = 1$ the branches of the hyperboloids are hyperbolic planes with constant curvature -1.

We also use the following notation. For the metric $\langle , \rangle$ on the fibres of E we set $\epsilon_1 = -1$ and $\epsilon_2 = \epsilon_3 = +1$. Further, denoting also by π the projection of E onto M, if x_i are local coordinates on M, set $q_i = x_i \circ \pi$. We will identify the tangent space of E at a point $x \in E$ with the fibre $E_{\pi(x)}$ through that point. For a section s of E we denote its vertical lift to E as a vector field by s^v (so $s^v = s \circ \pi$) and frequently utilise the natural identifications of J_a^v with J_a itself and with $\partial/\partial y_a$ in terms of the fibre coordinates y_1, y_2, y_3.

The Levi-Civita connection D of g gives rise to the horizontal lift X^h of a vector field X to the bundle E in the usual way:

$$X^h = \sum_i X^i \frac{\partial}{\partial q^i} - \sum_{a,b=1}^{3} \epsilon_b y^a (\langle D_X J_a, J_b \rangle \circ \pi) \frac{\partial}{\partial y_b}.$$

We now define two almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ on the hyperbolic twistor space $\mathcal{Z}$ as follows. Acting on horizontal vectors these are the same and given by $\mathcal{J}_1 X_\sigma^h = \mathcal{J}_2 X_\sigma^h = (jX)_\sigma^h$ where as above $j = \sum y_a J_a$ is the point $\sigma \in \mathcal{Z}$ considered as an endomorphism of TM. For a vertical vector tangent to $\mathcal{Z}$, $V = V^1(\partial/\partial y_1) + V^2(\partial/\partial y_2) + V^3(\partial/\partial y_3)$, let $\mathcal{J}_k V = (-1)^{k-1}\sigma \times V$, $k = 1, 2$, $\sigma \in \mathcal{Z}$, where $\times$ is the vector product determined by the paraquaternionic algebra. It is easy to check that h_t is Hermitian with respect to both $\mathcal{J}_1$ and $\mathcal{J}_2$.

The curvature operator $\mathcal{R} : \wedge^2 TM \to \wedge^2 TM$ admits an SO(2, 2)-irreducible decomposition

$$\mathcal{R} = \tfrac{1}{6}\tau I + \mathcal{B} + \mathcal{W}^+ + \mathcal{W}^-$$

similar to the four-dimensional Riemannian case. Here τ denotes the scalar curvature of the base manifold, $\mathcal{B}$ represents the traceless Ricci tensor and $\mathcal{W} = \mathcal{W}^+ + \mathcal{W}^-$ corresponds to the Weyl conformal curvature tensor. The metric g is said to be *self-dual* if $\mathcal{W}^- = 0$. Also it is often convenient to write the action of $\mathcal{R}$ on $\wedge^2 TM = \wedge^+ \oplus \wedge^-$ as

$$\mathcal{R} = \begin{pmatrix} \frac{1}{6}\tau I + \mathcal{W}^+ & \mathcal{B} \\ {}^*\mathcal{B} & \frac{1}{6}\tau I + \mathcal{W}^- \end{pmatrix}, \tag{2.1}$$

where we have made the standard identifications; e.g. of $\mathcal{B}$ with

$$\begin{pmatrix} 0 & \mathcal{B} \\ {}^*\mathcal{B} & 0 \end{pmatrix},$$

where ${}^*\mathcal{B}$ denotes the adjoint of the upper right hand block, $\mathcal{B}$.

The theory now develops as in the quaternionic Kähler case and we have the following result from [3] quite analogous to the classical twistor space theory.

Theorem 2.1. *On the hyperbolic twistor space $\mathcal{Z}$ of an oriented four-dimensional manifold M with a neutral metric g we have the following*:

(i) *The almost complex structure $\mathcal{J}_1$ is integrable if and only if the metric g is self-dual. The almost Hermitian structure $(\mathcal{J}_1, h_t)$ is semi-Kähler if and only if g is self-dual. It is indefinite Kähler if and only if the metric g is Einstein, self-dual, and $t\tau = -12$.*

(ii) *The almost complex structure $\mathcal{J}_2$ is never integrable. The almost Hermitian structure $(\mathcal{J}_2, h_t)$ is semi-Kähler if and only if g is self-dual. It is indefinite almost Kähler or nearly Kähler if and only if the metric g is Einstein, self-dual and $t\tau = 12$ or $t\tau = -6$, respectively*.

In the classical twistor space theory the almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ were introduced, respectively, by Atiyah et al. [1] and Eells and Salamon [4]. It is a result of Atiyah et al. that $\mathcal{J}_1$ is integrable if and only if the base manifold is self-dual [1]. Unlike $\mathcal{J}_1$, the almost complex structure $\mathcal{J}_2$ is never integrable as was observed by Eells and Salamon [4]. The Kähler condition for $(h_t, \mathcal{J}_1)$ in the classical case was studied by Friedrich and Kurke [6] who proved that $\mathcal{J}_1$ is Kähler if and only if the base manifold is Einstein, self-dual with $t\tau = 12$, $t > 0$. Note that the only compact Einstein, self-dual manifolds of positive scalar curvature are S^4 and $\mathbb{C}P^2$ with their canonical metrics [6,8].

3. Norm of the covariant derivative of the fundamental two-form

We denote by $\|\cdot\|_t$ the norm with respect to h_t and by $\|\cdot\|$ the norm with respect to g. Consider the almost Hermitian manifolds $(\mathcal{Z}, \mathcal{J}_k, h_t)$, $k = 1, 2$. The fundamental two-forms are defined by $\Omega_{k,t}(X, Y) = h_t(X, \mathcal{J}_k Y)$ $k = 1, 2$, but for simplicity we denote them by Ω. Similarly we denote the Levi-Civita connection of h_t simply by ∇.

We shall compute the norm of $\nabla\Omega$ in terms of the components of $\mathcal{R}$ in the decomposition (2.1).

Lemma 3.1. *The norm of* $\nabla\Omega$ *at* $\sigma \in \mathcal{Z}$ *is given by*

$$\|\nabla\Omega\|_t^2 = -\left[\frac{2}{t}\left(2+\frac{t\tau}{6}\right)^2 + \frac{t\tau^2}{18}(1+(-1)^k)\right] + \left[4+\frac{t\tau}{3}(2+(-1)^k)\right] g(\sigma, \mathcal{W}^-(\sigma))$$

$$t(3\|\mathcal{W}^-(\sigma)\|^2 - g(\sigma, \mathcal{W}^-(\sigma))^2 - 2\|\mathcal{W}^-\|^2) + t\left[\|\mathcal{B}(\sigma)\|^2 - \frac{\|\mathcal{B}\|^2}{2}\right] + (-1)^k t\,\mathrm{tr}(\mathcal{W}^- \circ S_\sigma)^2,$$

where S_σ *is the endomorphism of* $\wedge_p^-$, $p = \pi(\sigma)$, *defined by* $S_\sigma A = \sigma \times A$, $A \in \wedge_p^-$, *and* $\times$ *denotes the vector product determined by the paraquaternionic algebra.*

Proof. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ a local orthonormal frame on a neighbourhood of $p \in M$ such that $\|\mathbf{e}_1\| = \|\mathbf{e}_2\| = -\|\mathbf{e}_3\| = -\|\mathbf{e}_4\|$. As before we write $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -\varepsilon_4 = 1$ and we write ε_t for the sign of the non-zero number t. Let V be a vertical tangent vector such that $h_t(V, V) = \varepsilon_t$. Then $(\mathbf{e}_1^h, \mathbf{e}_2^h, \mathbf{e}_3^h, \mathbf{e}_4^h, V, \mathcal{J}_k V)$ is an orthonormal frame of $T_\sigma\mathcal{Z}$ and we have

$$\begin{aligned}\|\nabla\Omega\|_t^2 = &\sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b\varepsilon_t[(\nabla_V\Omega)(\mathbf{e}_a^h, \mathbf{e}_b^h)^2 + (\nabla_{\mathcal{J}_k V}\Omega)(\mathbf{e}_a^h, \mathbf{e}_b^h)^2] \\ &+2\sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b\varepsilon_t[(\nabla_{\mathbf{e}_a^h}\Omega)(\mathbf{e}_b^h, V)^2 + (\nabla_{\mathbf{e}_a^h}\Omega)(\mathbf{e}_b^h, \mathcal{J}_k V)^2]. \end{aligned} \tag{3.1}$$

For simplicity, we denote the complex structure K_σ on T_pM defined by σ by K. Then it is easy to check that

$$g(A, X \wedge KY) = \tfrac{1}{2} g(\sigma, A) g(X, Y) - g(\sigma \times A, X \wedge Y) \tag{3.2}$$

for any $A \in \wedge_p^-$ and $X, Y \in T_pM$. In particular

$$g(A, X \wedge KY + KX \wedge Y) = -2g(\sigma \times A, X \wedge Y). \tag{3.3}$$

Note that $X \wedge KY + KX \wedge Y \in \wedge_p^-$.

Now for $V \in \wedge_p^-$ we have $\mathcal{W}^+(V) = 0$, $\mathcal{W}^-(V) \in \wedge_p^-$ and $\mathcal{B}(V) \in \wedge_p^+$. Hence from Lemma 2 in [3] and (3.3) it follows that

$$\begin{aligned}
&\sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b(\nabla_V\Omega)(\mathbf{e}_a^h, \mathbf{e}_b^h)^2 \\
&= \sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b g\left(\left(2+\frac{t\tau}{6}\right)V - t\sigma\times\mathcal{W}^-(\sigma\times V), \mathbf{e}_a\wedge\mathbf{e}_b\right)^2 \\
&= \left(2+\frac{t\tau}{6}\right)^2\|V\|^2 + 2t\left(2+\frac{t\tau}{6}\right)g(\sigma\times V, \mathcal{W}^-(\sigma\times V)) + t^2\|\sigma\times\mathcal{W}^-(\sigma\times V)\|^2 \\
&= \left(2+\frac{t\tau}{6}\right)^2\|V\|^2 + 2t\left(2+\frac{t\tau}{6}\right)g(\sigma\times V, \mathcal{W}^-(\sigma\times V)) \\
&\quad + t^2\|\mathcal{W}^-(\sigma\times V)\|^2 - t^2 g(\sigma, \mathcal{W}^-(\sigma\times V))^2.
\end{aligned} \tag{3.4}$$

Replacing V by $\mathcal{J}_k V$ in (3.4) and adding to (3.4) we have

$$\begin{aligned}
\sum\nolimits_1 &:= \sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b((\nabla_V\Omega)(\mathbf{e}_a^h, \mathbf{e}_b^h)^2 + (\nabla_{\mathcal{J}_k V}\Omega)(\mathbf{e}_a^h, \mathbf{e}_b^h)^2) \\
&= 2\left(2+\frac{t\tau}{6}\right)^2\|V\|^2 + 2t\left(2+\frac{t\tau}{6}\right)(g(V, \mathcal{W}^-(V)) + g(\sigma\times V, \mathcal{W}^-(\sigma\times V))) \\
&\quad + t^2(\|\mathcal{W}^-(V)\|^2 + \|\mathcal{W}^-(\sigma\times V)\|^2) - t^2(g(\sigma, \mathcal{W}^-(V))^2 + g(\sigma, \mathcal{W}^-(\sigma\times V))^2).
\end{aligned} \tag{3.5}$$

Since $\mathcal{W}^-$ has vanishing trace,

$$g(\sigma, \mathcal{W}^-(\sigma)) - |t|g(V, \mathcal{W}^-(V)) - |t|g(\sigma\times V, \mathcal{W}^-(\sigma\times V)) = 0$$

and hence

$$g(V, \mathcal{W}^-(V)) + g(\sigma\times V, \mathcal{W}^-(\sigma\times V)) = \frac{\varepsilon_t}{t} g(\sigma, \mathcal{W}^-(\sigma)). \tag{3.6}$$

We also have

$$\|\mathcal{W}^-(V)\|^2 + \|\mathcal{W}^-(\sigma\times V)\|^2 = \frac{\varepsilon_t}{t}(\|\mathcal{W}^-(\sigma)\|^2 - \|\mathcal{W}^-\|^2), \tag{3.7}$$

and

$$g(\sigma, \mathcal{W}^-(V))^2 + g(\sigma, \mathcal{W}^-(\sigma\times V))^2 = \frac{\varepsilon_t}{t}(g(\sigma, \mathcal{W}^-(\sigma))^2 - \|\mathcal{W}^-(\sigma)\|^2). \tag{3.8}$$

Eqs. (3.5)–(3.8) now give

$$\begin{aligned}
\sum\nolimits_1 &= -\frac{2\varepsilon_t}{t}\left(2+\frac{t\tau}{6}\right)^2 + 2\varepsilon_t\left(2+\frac{t\tau}{6}\right)g(\sigma, \mathcal{W}^-(\sigma)) \\
&\quad + t\varepsilon_t(2\|\mathcal{W}^-(\sigma)\|^2 - \|\mathcal{W}^-\|^2 - g(\sigma, \mathcal{W}^-(\sigma))^2).
\end{aligned} \tag{3.9}$$

We now compute the second sum in (3.1). By Lemma 2 in [3] we get

$$\begin{aligned}
&\sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b(\nabla_{\mathbf{e}_a^h}\Omega)(\mathbf{e}_b^h, V)^2 \\
&\quad = \frac{t^2}{4}(\|\mathcal{R}(V)\|^2 + \|\mathcal{R}(\sigma\times V)\|^2) \\
&\qquad + \frac{t^2}{2}(-1)^k \sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b g(\mathcal{R}(V), \mathbf{e}_a\wedge\mathbf{e}_b) g(\mathcal{R}(\sigma\times V), \mathbf{e}_a\wedge K\mathbf{e}_b).
\end{aligned} \tag{3.10}$$

Now for $V\in\wedge_p^-$,

$$\|\mathcal{R}(V)\|^2 = \|\mathcal{B}(V)\|^2 + \tfrac{1}{36}\tau^2\|V\|^2 + \tfrac{1}{3}\tau g(V, \mathcal{W}^-(V)) + \|\mathcal{W}^-(V)\|^2, \tag{3.11}$$

and

$$\begin{aligned}
\|\mathcal{R}(\sigma\times V)\|^2 &= \|\mathcal{B}(\sigma\times V)\|^2 + \tfrac{1}{36}\tau^2\|V\|^2 + \tfrac{1}{3}\tau g(\sigma\times V, \mathcal{W}^-(\sigma\times V)) \\
&\quad + \|\mathcal{W}^-(\sigma\times V)\|^2.
\end{aligned} \tag{3.12}$$

Again for $V\in\wedge_p^-$, set $P(V) = (\tau/6)V + \mathcal{W}^-(V)$. Then $\mathcal{R}(V) = P(V) + \mathcal{B}(V)$ and we have

$$\begin{aligned}
&\sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b g(\mathcal{R}(V), \mathbf{e}_a\wedge\mathbf{e}_b) g(\mathcal{R}(\sigma\times V), \mathbf{e}_a\wedge K\mathbf{e}_b) \\
&\quad = \sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b g(P(V), \mathbf{e}_a\wedge\mathbf{e}_b) g(P(\sigma\times V), \mathbf{e}_a\wedge K\mathbf{e}_b) \\
&\qquad + \sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b g(P(V), \mathbf{e}_a\wedge\mathbf{e}_b) g(\mathcal{B}(\sigma\times V), \mathbf{e}_a\wedge K\mathbf{e}_b) \\
&\qquad + \sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b g(\mathcal{B}(V), \mathbf{e}_a\wedge\mathbf{e}_b) g(P(\sigma\times V), \mathbf{e}_a\wedge K\mathbf{e}_b) \\
&\qquad + \sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b g(\mathcal{B}(V), \mathbf{e}_a\wedge\mathbf{e}_b) g(\mathcal{B}(\sigma\times V), \mathbf{e}_a\wedge K\mathbf{e}_b).
\end{aligned}$$

Note that the latter three sums vanish because, for example,

$$\begin{aligned}
&\varepsilon_a\varepsilon_b g(P(V), \mathbf{e}_a\wedge\mathbf{e}_b) g(\mathcal{B}(\sigma\times V), \mathbf{e}_a\wedge K\mathbf{e}_b) \\
&\qquad + \varepsilon_b\varepsilon_a g(P(V), \mathbf{e}_b\wedge\mathbf{e}_a) g(\mathcal{B}(\sigma\times V), \mathbf{e}_b\wedge K\mathbf{e}_a) \\
&\quad = \varepsilon_a\varepsilon_b g(P(V), \mathbf{e}_a\wedge\mathbf{e}_b) g(\mathcal{B}(\sigma\times V), \mathbf{e}_a\wedge K\mathbf{e}_b + K\mathbf{e}_a\wedge\mathbf{e}_b) = 0
\end{aligned}$$

since $\mathbf{e}_a\wedge K\mathbf{e}_b + K\mathbf{e}_a\wedge\mathbf{e}_b \in \wedge_p^-$.

Now using (3.2) we get

$$\sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b g(P(V), \mathbf{e}_a\wedge\mathbf{e}_b) g(P(\sigma\times V), \mathbf{e}_a\wedge K\mathbf{e}_b)$$
$$= \sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b g(P(V), \mathbf{e}_a\wedge\mathbf{e}_b)\left[\frac{1}{2}g(\sigma, P(\sigma\times V))g(\mathbf{e}_a,\mathbf{e}_b)\right.$$
$$\left. -g(\sigma\times P(\sigma\times V), \mathbf{e}_a\wedge\mathbf{e}_b)\right] = -g(P(V), \sigma\times P(\sigma\times V))$$
$$= g(\sigma\times P(V), P(\sigma\times V)) = \frac{\tau^2}{36}\|V\|^2 + \frac{\tau}{6}[g(\sigma\times V, \mathcal{W}^-(\sigma\times V))$$
$$+ g(V, \mathcal{W}^-(V))] + g(\sigma\times\mathcal{W}^-(V), \mathcal{W}^-(\sigma\times V)).$$

Continuing using Eq. (3.6), this becomes

$$-\frac{\tau^2\varepsilon_t}{36t} + \frac{\tau\varepsilon_t}{6t} g(\sigma, \mathcal{W}^-(\sigma)) - g((\mathcal{W}^-\circ S_\sigma)^2(V), V).$$

Therefore

$$\sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b g(\mathcal{R}(V), \mathbf{e}_a\wedge\mathbf{e}_b) g(\mathcal{R}(\sigma\times V), \mathbf{e}_a\wedge K\mathbf{e}_b)$$
$$= -\frac{\tau^2\varepsilon_t}{36t} + \frac{\tau\varepsilon_t}{6t} g(\sigma, \mathcal{W}^-(\sigma)) - \frac{\varepsilon_t}{t} g((\mathcal{W}^-\circ S_\sigma)^2(V), V). \tag{3.13}$$

Now from Eqs. (3.10)–(3.13) we obtain

$$\sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b (\nabla_{\mathbf{e}_a^h}\Omega)(\mathbf{e}_b^h, V)^2$$
$$= -\frac{t\tau^2\varepsilon_t}{72}(1+(-1)^k) + \frac{t\tau\varepsilon_t}{12}(1+(-1)^k) g(\sigma, \mathcal{W}^-(\sigma))$$
$$+\frac{t\varepsilon_t}{4}(\|\mathcal{W}^-(\sigma)\|^2 - \|\mathcal{W}^-\|^2) + \frac{t\varepsilon_t}{4}\left(\|\mathcal{B}(\sigma)\|^2 - \frac{\|\mathcal{B}\|^2}{2}\right)$$
$$+(-1)^{k+1}\frac{t\varepsilon_t}{2} g((\mathcal{W}^-\circ S_\sigma)^2(V), V). \tag{3.14}$$

Applying (3.14) to $\mathcal{J}_k V$ and adding (3.14) we have

$$\sum\nolimits_2 := 2\sum_{1\le a,b\le 4} \varepsilon_a\varepsilon_b ((\nabla_{\mathbf{e}_a^h}\Omega)(\mathbf{e}_b^h, V)^2 + (\nabla_{\mathbf{e}_a^h}\Omega)(\mathbf{e}_b^h, \mathcal{J}_k V)^2)$$
$$= -\frac{t\tau^2\varepsilon_t}{18}(1+(-1)^k) + \frac{t\tau\varepsilon_t}{3}(1+(-1)^k) g(\sigma, \mathcal{W}^-(\sigma))$$
$$+t\varepsilon_t(\|\mathcal{W}^-(\sigma)\|^2 - \|\mathcal{W}^-\|^2) + t\varepsilon_t\left(\|\mathcal{B}(\sigma)\|^2 - \frac{\|\mathcal{B}\|^2}{2}\right)$$
$$+(-1)^k t\varepsilon_t \operatorname{tr}(\mathcal{W}^-\circ S_\sigma)^2 \tag{3.15}$$

since $(\mathcal{W}^- \circ S_\sigma)(\sigma) = 0$. Now since $\|\nabla\Omega\|_t^2 = \varepsilon_t(\sum_1 + \sum_2)$, the lemma follows from (3.9) and (3.15). □

4. Isotropic Kähler hyperbolic twistor spaces

We now prove the following theorem.

Theorem 4.1. *The hyperbolic twistor space* $(\mathcal{Z}, h_t, \mathcal{J}_k)$, $k = 1, 2$, *is isotropic Kähler if and only if* $k = 1$, $\mathcal{W}^- = 0$, $\mathcal{B}^2_{|\Lambda^-} = 0$ *and* $\tau t = -12$.

Proof. Setting

$$f(t) = \frac{1}{t}\left[4 + \frac{t\tau}{3}(2 + (-1)^k)\right]$$

$$h(t) = -2\|\mathcal{W}^-\|^2 - \frac{\|\mathcal{B}\|^2}{2} - \left[\frac{2}{t^2}\left(2 + \frac{t\tau}{6}\right)^2 + \frac{\tau^2}{18}(1 + (-1)^k)\right]$$

it follows that $\|\nabla\Omega\|_t^2 = 0$ if and only if

$$\begin{aligned} g(\sigma, \mathcal{W}^-(\sigma))^2 = {}& f(t)g(\sigma, \mathcal{W}^-(\sigma)) + 3\|\mathcal{W}^-(\sigma)\|^2 + \|\mathcal{B}(\sigma)\|^2 \\ & + (-1)^k \mathrm{tr}(\mathcal{W}^- \circ S_\sigma)^2 + h(t) \end{aligned} \tag{4.1}$$

for all $\sigma \in \mathcal{Z}$.

Let

$$D = \{(y_1, y_2, y_3) \in \mathbb{R}^3 | y_1^2 - y_2^2 - y_3^2 > 0\},$$

and $\|y\|^2 = y_1^2 - y_2^2 - y_3^2$. Then $(y_1/\|y\|, y_2/\|y\|, y_3/\|y\|)$ belongs to the hyperboloid $y_1^2 - y_2^2 - y_3^2 = 1$ and

$$\sigma = \frac{y_1}{\|y\|}s_1 + \frac{y_2}{\|y\|}s_2 + \frac{y_3}{\|y\|}s_3 \in \mathcal{Z}_p.$$

For simplicity denote by α_{ij} the inner product $g(s_i, \mathcal{W}^-(s_j))$. Then Eq. (4.1) can be written in the form

$$\left(\sum_{i,j=1}^{3} \alpha_{ij} y_i y_j\right)^2 = \|y\|^2 \left(\sum_{i,j=1}^{3} \gamma_{ij} y_i y_j\right)$$

for some coefficients γ_{ij}. Setting $y_3 = 0$ and comparing coefficients gives $\alpha_{11} = -\alpha_{22}$ and $\alpha_{12} = 0$. Similarly $\alpha_{11} = -\alpha_{33}$ and $\alpha_{13} = 0$. On the other hand $0 = \mathrm{tr}\,\mathcal{W}^- = \alpha_{11} - \alpha_{22} - \alpha_{33}$ which then gives $\alpha_{11} = \alpha_{22} = \alpha_{33} = 0$. Hence our equation takes the form

$$4\alpha_{23}^2 y_2^2 y_3^2 = (y_1^2 - y_2^2 - y_3^2)\left(\sum_{i,j=1}^{3} \gamma_{ij} y_i y_j\right),$$

which clearly implies that $\alpha_{23} = 0$. Thus $g(s_i, \mathcal{W}^-(s_j)) = 0$ for all $1 \le i, j \le 3$ giving that $\mathcal{W}^- = 0$.

Eq. (4.1) now takes the form

$$\|\mathcal{B}(\sigma)\|^2 = \tfrac{1}{2}\|\mathcal{B}\|^2 + g(t), \tag{4.2}$$

where

$$g(t) = \frac{2}{t^2}\left(2 + \frac{t\tau}{6}\right)^2 + \frac{\tau^2}{18}(1 + (-1)^k).$$

Note that $g(t) \ge 0$ with equality if and only if $k = 1$ and $t\tau = -12$.

Applying the same arguments as above we see that Eq. (4.2) is equivalent to $g(\mathcal{B}(s_i), \mathcal{B}(s_j)) = 0$ for $1 \le i \ne j \le 3$ and

$$\|\mathcal{B}(s_1)\|^2 = -\|\mathcal{B}(s_2)\|^2 = -\|\mathcal{B}(s_3)\|^2 = \tfrac{1}{2}\|\mathcal{B}\|^2 + g(t).$$

Since

$$\|\mathcal{B}(s_1)\|^2 - \|\mathcal{B}(s_2)\|^2 - \|\mathcal{B}(s_3)\|^2 = \tfrac{1}{2}\|\mathcal{B}\|^2,$$

we get

$$\|\mathcal{B}(s_1)\|^2 = -\|\mathcal{B}(s_2)\|^2 = -\|\mathcal{B}(s_3)\|^2 = -\tfrac{1}{2}g(t).$$

Suppose now that $g(t) > 0$ and note that if $\|x\|^2 \ge 0$ and $\|y\|^2 \ge 0$, we have $\langle x, y\rangle^2 \ge \|x\|^2\|y\|^2$. Then

$$0 = \langle \mathcal{B}(s_2), \mathcal{B}(s_3)\rangle^2 \ge \|\mathcal{B}(s_2)\|^2\|\mathcal{B}(s_3)\|^2 = (\tfrac{1}{2}g(t))^2,$$

a contradiction. Therefore $g(t) = 0$, i.e. $k = 1$, $\tau t = -12$ and $g(\mathcal{B}(s_i), \mathcal{B}(s_j)) = 0$ for all i, j. Finally since $\mathcal{B} : \wedge^2 TM \to \wedge^2 TM$ is a symmetric operator it follow that $\mathcal{B}^2_{|\Lambda^-} = 0$. □

5. Examples of neutral self-dual metrics with $\mathcal{B}^2 = 0$

To illustrate the phenomena of the theorem of the preceding section, we provide examples of neutral self-dual metrics with $\mathcal{B}^2 = 0$ for which $\mathcal{B} \ne 0$. We do not know of examples with $\mathcal{B}^2_{|\Lambda^-} = 0$ but $\mathcal{B}^2 \ne 0$. We give two classes of examples; in the first class we have $\mathcal{W} = 0$ and in the second class we have $\mathcal{W}^- = 0$ but $\mathcal{W}^+ \ne 0$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ defined by

$$[E_1, E_2] = \alpha E_2 + \beta E_3, \qquad [E_1, E_3] = \gamma E_2 + \delta E_3, \qquad [E_1, E_4] = pE_4,$$
$$[E_2, E_3] = qE_4, \qquad [E_2, E_4] = 0, \qquad [E_3, E_4] = 0$$

together with the condition $(\alpha + \delta - p)q = 0$ which gives the Jacobi identity. Define a neutral left-invariant metric g on G in terms of the dual basis $\{E^i\}$ by

$$g = E^1 \otimes E^1 + E^2 \otimes E^2 - E^3 \otimes E^3 - E^3 \otimes E^3.$$

It is now straightforward to compute the Levi-Civita connection and the curvature tensor of g. This in turn enables one to compute the curvature operator $\mathcal{R} : \wedge^2 TG \to \wedge^2 TG$ with respect to the bases $\{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$ and $\{s_1, s_2, s_3\}$ of $\wedge^+$ and $\wedge^-$, respectively. For the study of $\mathcal{B}$ we need:

$$
\begin{aligned}
&g(\mathcal{R}(s_1), \bar{s}_1) = \tfrac{1}{4}(\beta^2 - \gamma^2) + \tfrac{1}{2}(\beta(\beta - \gamma)) + p\delta - \alpha^2 - \tfrac{1}{4}q^2,\\
&g(\mathcal{R}(s_1), \bar{s}_2) = \beta\delta - \alpha\gamma + \tfrac{1}{2}p(\beta - \gamma), \qquad g(\mathcal{R}(s_1), \bar{s}_3) = 0,\\
&g(\mathcal{R}(s_2), \bar{s}_1) = \beta\delta - \alpha\gamma + \tfrac{1}{2}p(\beta - \gamma),\\
&g(\mathcal{R}(s_2), \bar{s}_2) = \tfrac{1}{4}(\beta^2 - \gamma^2) + \tfrac{1}{2}(\gamma(\beta - \gamma)) + \delta^2 - p\alpha + \tfrac{1}{4}q^2,\\
&g(\mathcal{R}(s_2), \bar{s}_3) = 0, \qquad g(\mathcal{R}(s_3), \bar{s}_1) = 0,\\
&g(\mathcal{R}(s_3), \bar{s}_2) = 0, \qquad g(\mathcal{R}(s_3), \bar{s}_3) = p^2 - \tfrac{3}{4}q^2 - \tfrac{1}{4}(\beta - \gamma)^2 - \alpha\delta.
\end{aligned}
$$

For the study of $\mathcal{W}^-$ we need:

$$
\begin{aligned}
&g(\mathcal{R}(\bar{s}_1), \bar{s}_1) = \tfrac{1}{4}(\beta^2 - \gamma^2) + \tfrac{1}{2}\beta(\beta - \gamma) - \alpha q + \tfrac{1}{4}q^2 - \alpha^2 - p\delta,\\
&g(\mathcal{R}(\bar{s}_1), \bar{s}_2) = \beta\delta - \alpha\gamma + \tfrac{1}{2}(\beta - \gamma)q - \tfrac{1}{2}p(\beta - \gamma), \qquad g(\mathcal{R}(\bar{s}_1), \bar{s}_3) = 0,\\
&g(\mathcal{R}(\bar{s}_2), \bar{s}_2) = \tfrac{1}{4}(\beta^2 - \gamma^2) + \tfrac{1}{2}\gamma(\beta - \gamma) + \delta^2 + p\alpha + \delta q - \tfrac{1}{4}q^2,\\
&g(\mathcal{R}(\bar{s}_2), \bar{s}_3) = 0, \qquad g(\mathcal{R}(\bar{s}_3), \bar{s}_3) = p^2 - pq + \tfrac{3}{4}q^2 + \tfrac{1}{4}(\beta - \gamma)^2 + \alpha\delta.
\end{aligned}
$$

For the study of $\mathcal{W}^+$we need:

$$
\begin{aligned}
&g(\mathcal{R}(s_1), s_1) = \tfrac{1}{4}(\beta^2 - \gamma^2) + \tfrac{1}{2}\beta(\beta - \gamma) + \alpha q + \tfrac{1}{4}q^2 - \alpha^2 - p\delta,\\
&g(\mathcal{R}(s_1), s_2) = \beta\delta - \alpha\gamma - \tfrac{1}{2}(\beta - \gamma)q - \tfrac{1}{2}p(\beta - \gamma), \qquad g(\mathcal{R}(s_1), s_3) = 0,\\
&g(\mathcal{R}(s_2), s_2) = \tfrac{1}{4}(\beta^2 - \gamma^2) + \tfrac{1}{2}\gamma(\beta - \gamma) + \delta^2 + p\alpha - \delta q - \tfrac{1}{4}q^2,\\
&g(\mathcal{R}(s_2), s_3) = 0, \qquad g(\mathcal{R}(s_3), s_3) = p^2 + pq + \tfrac{3}{4}q^2 ** + \tfrac{1}{4}(\beta - \gamma)^2 + \alpha\delta.
\end{aligned}
$$

We again recall the decomposition of the curvature operator

$$\mathcal{R} = \tfrac{1}{6}\tau I + \mathcal{B} + \mathcal{W}^+ + \mathcal{W}^-$$

written as

$$\mathcal{R} = \begin{pmatrix} \frac{1}{6}\tau I + \mathcal{W}^+ & \mathcal{B} \\ {}^*\mathcal{B} & \frac{1}{6}\tau I + \mathcal{W}^- \end{pmatrix},$$

where we have made the standard identifications, especially of $\mathcal{B}$ with

$$\begin{pmatrix} 0 & \mathcal{B} \\ {}^*\mathcal{B} & 0 \end{pmatrix}.$$

Setting $b_{ji} = g(\mathcal{R}(s_i), \bar{s}_j)$ and recalling that the induced metrics on $\wedge^-$ and $\wedge^+$ have signature $(+, -, -)$, the matrices for $\mathcal{B}$ and its adjoint ${}^*\mathcal{B}$ are

$$\begin{pmatrix} b_{11} & b_{12} & 0 \\ -b_{21} & -b_{22} & 0 \\ 0 & 0 & -b_{33} \end{pmatrix}, \qquad \begin{pmatrix} b_{11} & b_{21} & 0 \\ -b_{12} & -b_{22} & 0 \\ 0 & 0 & -b_{33} \end{pmatrix}.$$

Thus the condition for $\mathcal{B}^2_{|\Lambda^-}$ to vanish becomes

$$\begin{pmatrix} b_{11}^2 - b_{21}^2 & b_{11}b_{12} - b_{21}b_{22} & 0 \\ -b_{11}b_{12} + b_{21}b_{22} & b_{22}^2 - b_{12}^2 & 0 \\ 0 & 0 & b_{33}^2 \end{pmatrix} = 0.$$

From here we see that $\mathcal{B}^2_{|\Lambda^-}$ will vanish if and only if

$$g(\mathcal{R}(s_1), \bar{s}_1) = \epsilon g(\mathcal{R}(s_1), \bar{s}_2), \quad g(\mathcal{R}(s_2), \bar{s}_2) = \epsilon g(\mathcal{R}(s_2), \bar{s}_1), \quad g(\mathcal{R}(s_3), \bar{s}_3) = 0,$$

where $\epsilon = \pm 1$. Similarly we see that self-duality becomes

$$g(\mathcal{R}(s_1), s_2) = 0, \quad g(\mathcal{R}(s_1), s_1) = -g(\mathcal{R}(s_2), s_2) = -g(\mathcal{R}(s_3), s_3).$$

Since the Jacobi identity leads to the condition $(\alpha + \delta - p)q = 0$, we first consider the case $q = 0$. We then have the following equations

$$\tfrac{1}{4}(\beta^2 - \gamma^2) + \tfrac{1}{2}\beta(\beta - \gamma) + p\delta - \alpha^2 = \epsilon(\beta\delta - \alpha\gamma + \tfrac{1}{2}p(\beta - \gamma)), \tag{5.1}$$

$$\tfrac{1}{4}(\beta^2 - \gamma^2) + \tfrac{1}{2}\gamma(\beta - \gamma) + \delta^2 - p\alpha = \epsilon(\beta\delta - \alpha\gamma + \tfrac{1}{2}p(\beta - \gamma)), \tag{5.2}$$

$$p^2 - \tfrac{1}{4}(\beta - \gamma)^2 - \alpha\delta = 0, \tag{5.3}$$

$$\beta\delta - \alpha\gamma - \tfrac{1}{2}p(\beta - \gamma) = 0, \tag{5.4}$$

$$\beta^2 - \gamma^2 = \alpha^2 - p\alpha + p\delta - \delta^2, \tag{5.5}$$

$$\beta(\beta - \gamma) = \alpha^2 + p\delta - p^2 - \alpha\delta. \tag{5.6}$$

Eqs. (5.5) and (5.6) readily yield

$$\gamma(\beta - \gamma) = p^2 + \alpha\delta - p\alpha - \delta^2.$$

Substituting this and (5.6) into (5.3) we have

$$0 = \tfrac{1}{4}(3p + \alpha + \delta)(2p - \alpha - \delta).$$

The case $p = -(1/3)(\alpha+\delta)$ leads to a contradiction and we study the case $p = (1/2)(\alpha+\delta)$. If $p = 0$, the system is easy to solve and gives $\mathcal{B} = 0$. For $p \neq 0$, introduce a parameter x by $\alpha = xp$ and then $\delta = (2 - x)p$. Adding (5.1) and (5.2), and using (5.4), we get

$$\beta^2 - \gamma^2 + 6p^2 - 6xp^2 = 2\epsilon p(\beta - \gamma),$$

which upon comparing with (5.5) yields

$$\beta - \gamma = 2\epsilon p(1 - x).$$

On the other hand, with $\alpha = xp$ and $\delta = (2 - x)p$, Eq. (5.4) is

$$(\tfrac{3}{2} - x)\beta + (\tfrac{1}{2} - x)\gamma = 0.$$

Therefore

$$\alpha = xp, \qquad \delta = (2-x)p, \qquad \beta = \epsilon p(\tfrac{1}{2} - x), \qquad \gamma = \epsilon p(x - \tfrac{3}{2}),$$

and one can easily check that these satisfy Eqs. (5.1)–(5.6). Thus, with respect to the basis $\{\bar{s}_1, \bar{s}_2, \bar{s}_3\, s_1, s_2, s_3\}$ of $\wedge^2 TG = \wedge^+ \oplus \wedge^-$, the curvature operator is given by

$$\mathcal{R} = p^2 \begin{pmatrix} -2 & 0 & 0 & 2(1-x) & 2\epsilon(1-x) & 0 \\ 0 & -2 & 0 & -2\epsilon(1-x) & -2(1-x) & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 2(1-x) & 2\epsilon(1-x) & 0 & -2 & 0 & 0 \\ -2\epsilon(1-x) & -2(1-x) & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

Therefore the metric is self-dual and, for $x \neq 1$, we have $\mathcal{B}^2 = 0$ but $\mathcal{B} \neq 0$. Note also that we can also consider p as a parameter in the Lie algebras, so considering the Lie algebras and metric together we have a two-parameter family of examples. Finally the scalar curvature in these examples is $-12p^2$ and taking $t = 1/p^2$ we have that the hyperbolic twistor space $(\mathcal{Z}, h_{1/p^2}, \mathcal{J}_1)$, is isotropic Kähler but not indefinite Kähler.

We now turn to cases where $\alpha + \delta - p = 0$ but $q \neq 0$; we also assume $p \neq 0$. The equations for $\mathcal{B}^2_{|\Lambda^-}$ are

$$\tfrac{1}{4}(\beta^2 - \gamma^2) + \tfrac{1}{2}\beta(\beta - \gamma) + \alpha\delta + \delta^2 - \alpha^2 - \tfrac{1}{4}q^2 = \epsilon(\tfrac{3}{2}\beta\delta - \tfrac{3}{2}\alpha\gamma + \tfrac{1}{2}\alpha\beta - \tfrac{1}{2}\delta\gamma), \tag{5.7}$$

$$\tfrac{1}{4}(\beta^2 - \gamma^2) + \tfrac{1}{2}\gamma(\beta - \gamma) + \delta^2 - \alpha^2 - \alpha\delta + \tfrac{1}{4}q^2 = \epsilon(\tfrac{3}{2}\beta\delta - \tfrac{3}{2}\alpha\gamma + \tfrac{1}{2}\alpha\beta - \tfrac{1}{2}\delta\gamma), \tag{5.8}$$

$$\alpha^2 + \alpha\delta + \delta^2 - \tfrac{3}{4}q^2 - \tfrac{1}{4}(\beta - \gamma)^2 = 0, \tag{5.9}$$

and those dealing with self-duality are

$$\beta\delta - \alpha\gamma - \beta q + \gamma q - \alpha\beta + \delta\gamma = 0, \tag{5.10}$$

$$\beta^2 - \gamma^2 = -\alpha q + \delta q, \tag{5.11}$$

$$\beta(\beta - \gamma) = -2\alpha\delta - q^2 - 2\alpha q - \delta q. \tag{5.12}$$

Eqs. (5.11) and (5.12) readily give

$$\gamma(\beta - \gamma) = 2\alpha\delta + q^2 + \alpha q + 2\delta q, \tag{5.13}$$

and then subtracting (5.13) from (5.12) gives

$$(\beta - \gamma)^2 = -2q^2 - 3(\alpha + \delta)q - 4\alpha\delta. \tag{5.14}$$

Substituting (5.9) into the difference of (5.7) and (5.8) we have $q = \pm(\alpha + \delta)$ and returning to (5.9), $(\alpha - \delta)^2 = (\beta - \gamma)^2$. If $q = \alpha + \delta$, (5.14) readily gives $q = 0$, so we only consider $q = -\alpha - \delta$.

If $q = -\alpha - \delta$ and $\alpha - \delta = \beta - \gamma$, Eq. (5.10) gives $\beta(\alpha - \delta) = \alpha(\alpha - \delta)$. The case $\alpha = \delta$ leads to $\mathcal{B} = 0$. If $\alpha \neq \delta$, $\beta = \alpha$ and therefore $\gamma = \delta$. One can now check that the equations for $\mathcal{B}^2_{|\Lambda^-} = 0$ and self-duality are satisfied with $\epsilon = -1$. The matrix of the curvature operator is

$$\mathcal{R} = \begin{pmatrix} \beta^2-\gamma^2 & \gamma^2-\beta^2 & 0 & \frac{1}{2}(\gamma^2-\beta^2) & \frac{1}{2}(\beta^2-\gamma^2) & 0 \\ \beta^2-\gamma^2 & \gamma^2-\beta^2 & 0 & \frac{1}{2}(\gamma^2-\beta^2) & \frac{1}{2}(\beta^2-\gamma^2) & 0 \\ 0 & 0 & -3(\beta+\gamma)^2 & 0 & 0 & 0 \\ \frac{1}{2}(\gamma^2-\beta^2) & \frac{1}{2}(\beta^2-\gamma^2) & 0 & -(\beta+\gamma)^2 & 0 & 0 \\ \frac{1}{2}(\gamma^2-\beta^2) & \frac{1}{2}(\beta^2-\gamma^2) & 0 & 0 & -(\beta+\gamma)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\beta+\gamma)^2 \end{pmatrix},$$

and the matrix for $\mathcal{W}^+$ is

$$\mathcal{W}^+ = \begin{pmatrix} 2\beta(\beta+\gamma) & \gamma^2-\beta^2 & 0 \\ \beta^2-\gamma^2 & 2\gamma(\beta+\gamma) & 0 \\ 0 & 0 & -2(\beta+\gamma)^2 \end{pmatrix}.$$

Thus we have a two-parameter family of examples with $\mathcal{B}^2 = 0$ and $\mathcal{W} \neq 0$.

Similarly $q = -\alpha - \delta$ and $\alpha - \delta = \gamma - \beta$ leads to $\beta = -\alpha$, $\gamma = -\delta$ and $\epsilon = 1$. The resulting curvature operator is

$$\mathcal{R} = \begin{pmatrix} \beta^2-\gamma^2 & \beta^2-\gamma^2 & 0 & \frac{1}{2}(\gamma^2-\beta^2) & \frac{1}{2}(\gamma^2-\beta^2) & 0 \\ \gamma^2-\beta^2 & \gamma^2-\beta^2 & 0 & \frac{1}{2}(\beta^2-\gamma^2) & \frac{1}{2}(\beta^2-\gamma^2) & 0 \\ 0 & 0 & -3(\beta+\gamma)^2 & 0 & 0 & 0 \\ \frac{1}{2}(\gamma^2-\beta^2) & \frac{1}{2}(\gamma^2-\beta^2) & 0 & -(\beta+\gamma)^2 & 0 & 0 \\ \frac{1}{2}(\beta^2-\gamma^2) & \frac{1}{2}(\beta^2-\gamma^2) & 0 & 0 & -(\beta+\gamma)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\beta+\gamma)^2 \end{pmatrix}$$

again giving a two-parameter family of examples with $\mathcal{B}^2 = 0$ and $\mathcal{W} \neq 0$.

In the last two examples the scalar curvature is $\tau = -6(\beta + \gamma)^2$ and the corresponding hyperbolic twistor space $(\mathcal{Z}, h_t, \mathcal{J}_1)$ for $t = 2/(\beta + \gamma)^2$ is a non-Kähler isotropic Kähler manifold.

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